PHYS 798C Spring 2022
Lecture 2 Summary

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The heart and soul of superconductivity is the Meissner Effect. This feature uniquely distinguishes superconductivity from many other states of matter. Here we discuss some simple phenomenological approaches to describing the Meissner effect quantitatively. The arguments here are not rigorous or microscopic in nature. However, they lead to important insights about the nature of the Meissner state, and help to build intuition about superconductors.

I. THE LONDON EQUATIONS

The brothers F. and H. London wrote down two simple equations which conveniently incorporate the electrodynamic response of a superconductor. These equations describe the microscopic electric ($\vec{E}$) and magnetic ($\vec{B}$) fields inside a superconductor.

To derive the first London equation, think of the net force acting on the charge carrier in a normal metal (Drude model):

$$\frac{d(m\vec{v})}{dt} = e\vec{E} - \frac{m\vec{v}}{\tau}$$

(Note that this is a LOCAL equation. It assumes that only the local electric field influences the drift velocity. As such, it requires the mean free path be less than the magnetic penetration depth, $\ell_{mfp} = v_F\tau < \lambda_L$). Here $\vec{v}$ is the velocity of the charge carrier of charge e, m is its mass, $\vec{E}$ is the local electric field, v_F is the Fermi velocity, and τ is a phenomenological scattering time for the carrier which describes how long it takes the scattering to bring the velocity of the carrier to zero. In a normal metal in steady state, the drift velocity achieves a constant value, meaning that the electric force and scattering forces balance, leading to:

$$\langle\vec{v}\rangle = e\vec{E}\tau/m.$$

Here $\langle\vec{v}\rangle$ is the average or "drift" velocity of the charge carrier. If there are n carriers per unit volume, the electrical current density can be written as $\vec{J} = ne\langle\vec{v}\rangle$, so

$$\vec{J} = \frac{ne^2\tau}{m}\vec{E},$$

which is Ohm's Law ($\vec{J} = \sigma\vec{E}$) with the conductivity $\sigma = \frac{ne^2\tau}{m}$. Ohm's law says that an electric current in a normal metal is a consequence of an applied electric field.

A. First London Equation

To model a superconductor, we shall suppose that there is a density of superconducting electrons, n_s, and they do not have their velocities reduced to zero by means of scattering. (See Tinkham p. 5 for why $\tau \to \infty$ does not give perfect conductivity) From the above equation, this means that the electrons will accelerate in an applied electric field! $m\partial\vec{v}/\partial t = e\vec{E}$, giving rise to the **first London equation**:

$$\frac{\partial\vec{J}_s}{\partial t} = \frac{n_s e^2}{m}\vec{E} \text{ or } \frac{\partial(\Lambda\vec{J}_s)}{\partial t} = \vec{E}$$

where $\Lambda := m/(n_s e^2)$. Strictly speaking, this equation only holds for ac currents and electric fields, since it predicts diverging currents for large times at dc. The first London equation says that in order to create an alternating current (i.e. a non-zero $\partial\vec{J}_s/\partial t$) it is necessary to establish an electric field in the superconductor. This has implications for the finite-frequency losses in superconductors. If any un-paired electrons (quasiparticles) are around, they will be accelerated by the electric field and cause Ohmic dissipation. Hence a superconductor has a small but finite dissipation when illuminated with a finite frequency electromagnetic wave at temperatures above zero Kelvin. Superconductors are only dissipation-less at zero frequency, or at finite frequency at zero temperature (for a fully-gapped superconductor).

We define a new quantity, the London penetration depth λ_L as,

$$\frac{\partial \vec{J}_s}{\partial t} = \frac{1}{\Lambda}\vec{E} = \frac{1}{\mu_0 \lambda_L^2}\vec{E}$$

where $\Lambda := \mu_0 \lambda_L^2 = m/(n_s e^2)$. The London magnetic penetration depth is an important new length scale, λ_L. It is defined as,

$$\lambda_L = \sqrt{\frac{m}{\mu_0 n_s e^2}}$$

To get a deeper insight into the first London Equation and this new length scale, start with the Maxwell equation for the microscopic fields (Ampere's Law),

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_s + \mu_0 \frac{\partial \vec{D}}{\partial t}$$

and ignoring the displacement current (this is usually appropriate in superconductors because we often consider only frequencies $\omega < 2\Delta \sim$ THz roughly), take the time derivative of both sides and use the first London equation, to obtain,

$$\vec{\nabla} \times \frac{\partial \vec{B}}{\partial t} = \mu_0 \frac{\partial \vec{J}_s}{\partial t} = \frac{1}{\lambda_L^2}\vec{E}$$

Now take the curl of both sides,

$$\vec{\nabla} \times \vec{\nabla} \times \frac{\partial \vec{B}}{\partial t} = \frac{1}{\lambda_L^2}\vec{\nabla} \times \vec{E}$$

The electric field curls around the time-varying magnetic field (Faraday's law)

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

to get

$$\vec{\nabla} \times \vec{\nabla} \times \frac{\partial \vec{B}}{\partial t} = -\frac{1}{\lambda_L^2}\frac{\partial \vec{B}}{\partial t}$$

Integrating both sides with respect to time yields

$$\vec{\nabla} \times \vec{\nabla} \times \vec{B} + \frac{1}{\lambda_L^2}\vec{B} = 0$$

Now use the vector identity

$$\vec{\nabla} \times \vec{\nabla} \times \vec{B} = \vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B}$$

And the fact that $\vec{\nabla} \cdot \vec{B} = 0$ to arrive at

$$\nabla^2 \vec{B} = \frac{1}{\lambda_L^2}\vec{B}$$

This equation admits solutions of the general form $B(x) = B_0 e^{\pm x/\lambda_L}$, so the London penetration depth λ_L represents the exponential screening length of the magnetic field in the superconductor. This length scale is also commonly referred to as the "magnetic penetration depth" for obvious reasons. This equation shows that the magnetic field is excluded from the bulk of a superconductor (B is effectively zero once you are 5-10 penetration depths from the surface), and describes the result of the Meissner effect. A similar result can be derived for the electric field:

$$\nabla^2 \vec{E} = \frac{1}{\lambda_L^2} \vec{E}$$

showing that it is screened on the same length scale. Finally, since $\vec{J}_s \propto \vec{\nabla} \times \vec{B}$ we see that $J_s \propto e^{\pm x/\lambda_L}$ too, hence the screening currents that establish the Meissner state flow within a few penetration depths of the surface. The London penetration depth can be estimated for Al, which has a total carrier density of $n = 18.1 \times 10^{22}$ free electrons/cm^3, where we find $\lambda_L = 12.5$ nm. This is an important microscopic length scale in superconductors.

B. The Second London equation

Start with London's first equation, $\frac{\partial \vec{J}_s}{\partial t} = \frac{n_s e^2}{m} \vec{E}$ and take the curl of both sides: $\frac{\partial}{\partial t} \vec{\nabla} \times \vec{J}_s = \frac{n_s e^2}{m} \vec{\nabla} \times \vec{E}$ and use Faraday's law ($\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$) to write $\frac{\partial}{\partial t}[\vec{\nabla} \times \vec{J}_s + \frac{n_s e^2}{m} \vec{B}] = 0$. This is a combination of Faraday's law and Lenz's law. It says that a conductor will develop a current to oppose a change in flux in the material. This equation does not predict the Meissner effect of course, only diamagnetic response to time-varying fields. The London's thought that superconductors could be described by simply making the square-bracketed term equal to zero:

$\vec{\nabla} \times \vec{J}_s = -\frac{n_s e^2}{m} \vec{B}$ (the London hypothesis)
or, $\Lambda \vec{\nabla} \times \vec{J}_s = -\vec{B}$ (**the second London equation**).

This is not a rigorous derivation of the second London equation. de Gennes, for example, uses an energy minimization argument to arrive at the same result. See P. G. de Gennes, *Superconductivity of Metals and Alloys.* (Addison-Wesley Publishing Co., Redwood City, CA, 1989).

This equation states that a magnetic field applied to a superconductor creates a screening current, such that the curl of that current is oppositely directed to the field. It says that dc currents are controlled by magnetic fields, as opposed to normal metals where they are controlled by electric fields, $\vec{J}_n = \sigma \vec{E}$.

This equation only applies under limited conditions:

1. Applied magnetic field H must be "small" and treated as a perturbation, namely $H \ll H_c$,

2. The superfluid density n_s should be uniform in space and time.

3. Local electrodynamics, namely $\lambda_L \gg \frac{1}{1/\xi_0 + 1/\ell_{mfp}}$.

C. The London Gauge

One can derive both London equations by starting with the following simple expression relating the vector potential to the super-current response, $\vec{J}_s = -\frac{1}{\Lambda} \vec{A}$, where $\vec{A}$ is the vector potential. (Spoiler alert: this relation turns out to be missing one very important term, but it works for now.) Taking the time-derivative of both sides yields London's first equation, while taking the curl of both sides yields the second London equation. However one must choose an appropriate gauge for the vector potential, and the standard convention is the London gauge:

a) $\Lambda \vec{J}_s = -\vec{A}$ is obeyed on all surfaces and in the bulk of the superconductor,

b) $\vec{\nabla} \cdot \vec{A} = \vec{\nabla} \cdot \vec{J}_s = 0$,

c) $\vec{A} \to 0$ deep (many penetration depths) inside the superconductor.

These conditions can break down in a number of situations, including near a superconductor/normal boundary in a current-carrying wire where the divergence condition on $\vec{J}_s$ is not satisfied, or in a multiply-connected superconductor where the superconducting order parameter can no longer be taken to be purely real.

D. Comparing Ohm's Law and the London Equations

To induce a dc current in a normal metal, you apply a dc *electric* field. To induce a dc current in a superconductor, you apply a dc *magnetic* field! We see from London's first equation that an electric field governs the *acceleration* of the supercurrent. The infinite conductivity of a superconductor at zero frequency (discussed in the next lecture) means that a superconductor cannot support a dc electric field. The equation also says that an ac electric field is required to establish and support an alternating current in a superconductor. This ac electric field will also be witnessed by any co-existing normal electrons, due to the finite inertia of the superconducting electrons, creating dissipation. Below is a table summarizing the (somewhat over-)simplified constitutive equations for superconductors and normal metals, in the local limit.

Superconductor	Normal Metal
$\frac{\partial(\Lambda \vec{J}_s)}{\partial t} = \vec{E}$	$\vec{J}_n = \sigma \vec{E}$
$\vec{\nabla} \times (\Lambda \vec{J}_s) = -\vec{B}$	$\vec{\nabla} \times (\frac{\vec{J}_n}{\sigma}) = -\frac{\partial \vec{B}}{\partial t}$

An electric field governs the acceleration of the superfluid, whereas the same electric field simply produces a current in a normal metal. Also, a dc magnetic field creates a screening supercurrent, whereas a time-dependent magnetic field is required to create a current in a normal metal. Note that the response of the superconductor to $\vec{E}$ and $\vec{B}$ is basically in quadrature (90 degrees out of phase) to the response of a normal metal. This has interesting ramifications for the electrodynamics of superconductors.

For a superconductor in the Meissner state, screening currents only flow within a few magnetic penetration depths of the surface of the conductor. The currents are diamagnetic in nature, meaning that they create a self-field that opposes the external field. Another way to say it is that a current-carrying superconductor *in the Meissner state* will push all of the dc and ac currents to its surfaces and exclude them from the bulk. The class web site shows examples of screening currents in a superconducting cylinder and a strip.